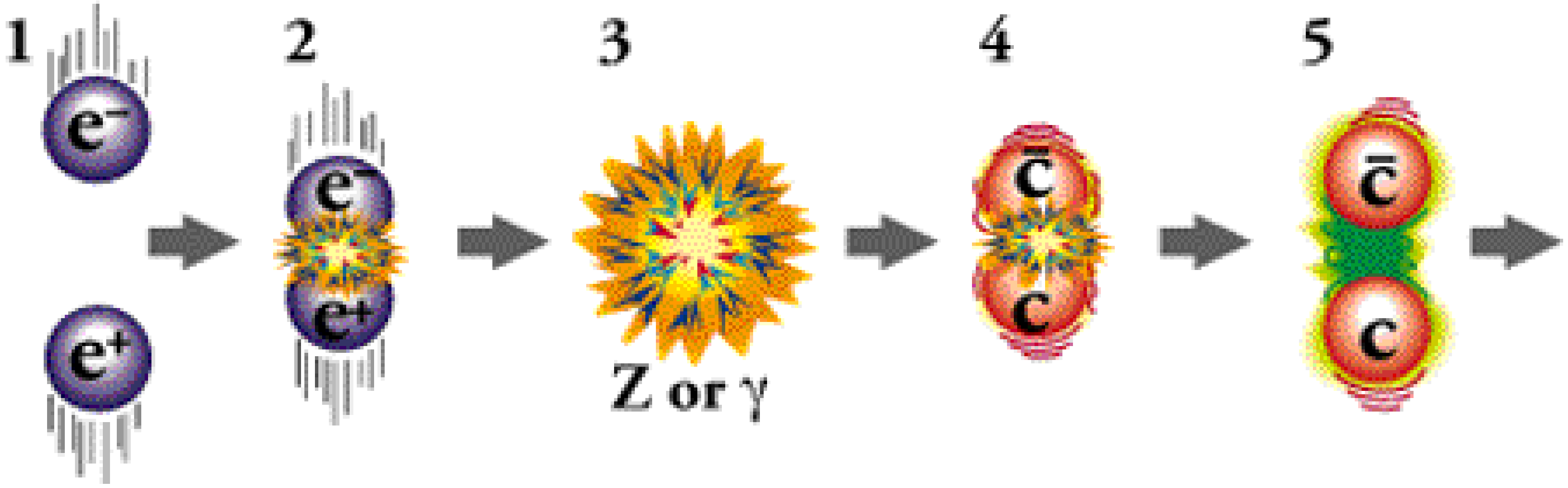


# Particle Physics I

## **Lecture 8:** Electron-positron annihilation

Prof. Radoslav Marchevski
November 5th 2025

# Short recap and learning targets

- **Ultimate goal:** derivation of the cross section for $e^+e^- \rightarrow \mu^+\mu^-$ annihilation as an example of calculation in QED

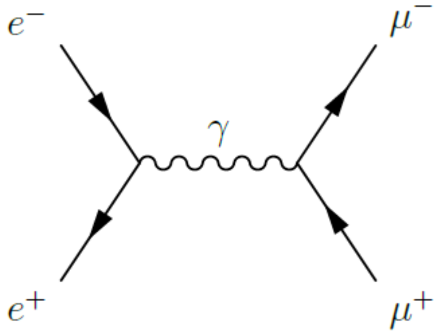
Learning targets

- basic ingredients of a QED calculation
- electron probability current
- muon probability current
- matrix element calculation using Feynman rules
- compute the cross section for $e^+e^- \rightarrow \mu^+\mu^-$ annihilation process taking into account the helicity of the electrons and muons

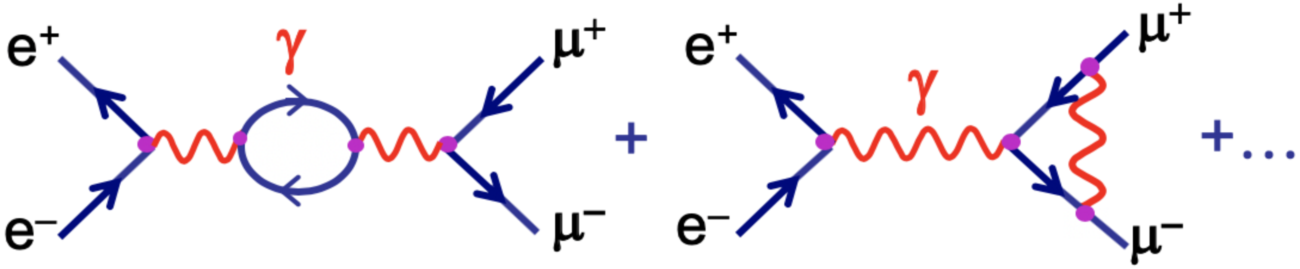
# QED calculations

How to calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

1. Draw all possible Feynman diagrams
   - for $e^+e^- \to \mu^+\mu^-$ there is only one lowest order diagram: $M \propto e^2 \propto \alpha_{\text{em}}$



   - plus many second order diagrams: $M \propto e^4 \propto \alpha_{\text{em}}^2$



2. For each diagram, calculate the matrix element using Feynman rules

# QED calculations

How to calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

3. Sum the individual matrix elements (i.e. sum the amplitudes)

   - $M_{fi} = M_1 + M_2 + M_3 + \cdots$
   - **note:** summing amplitudes ⟹ different diagrams can interfere either positively or negatively!

4. Square the matrix element $\left|M_{fi}\right|^2 = (M_1^* + M_2^* + M_3^* + \cdots)(M_1 + M_2 + M_3 + \cdots)$

   - this gives the full perturbative expansion in $\alpha_{\mathrm{em}}$

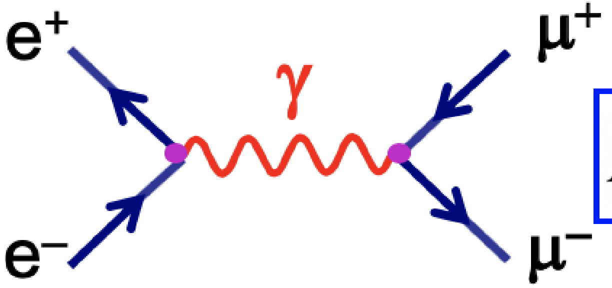
**Question:** By which factor are higher-order corrections suppressed?

# QED calculations

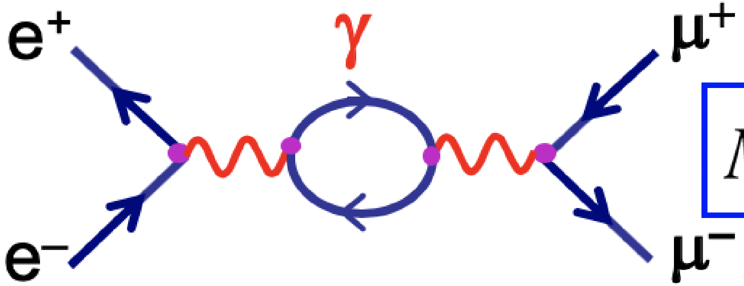
How to calculate a cross section using QED (e.g. $e^+e^- \rightarrow \mu^+\mu^-$):

3. Sum the individual matrix elements (i.e. sum the amplitudes)
   - $M_{fi} = M_1 + M_2 + M_3 + \cdots$
   - **note:** summing amplitudes ⟹ different diagrams can interfere either positively or negatively!

4. Square the matrix element $\left|M_{fi}\right|^2 = (M_1^* + M_2^* + M_3^* + \cdots)(M_1 + M_2 + M_3 + \cdots)$
   - this gives the full perturbative expansion in $\alpha_{\text{em}}$
   - for QED: $\alpha_{\text{em}} \sim 1/137$ and the lowest order diagram dominates and for most purposes it is sufficient to **neglect** higher order diagrams



$$\boxed{M^2 \propto \alpha_{em}^2}$$



$$\boxed{M^2 \propto \alpha_{em}^4}$$

   - interference term suppressed by $\alpha_{\text{em}}$ ⟹ leading–order QED calculations correct to ≈ 1%

# QED calculations

How to calculate a cross section using QED (e.g. $e^+e^- \rightarrow \mu^+\mu^-$):

5. Calculate the decay rate/cross section using the previous formulae

   - for a decay with center-of-mass frame momentum $p^*$ of the final-state particles:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int \left|M_{fi}\right|^2 d\Omega$$

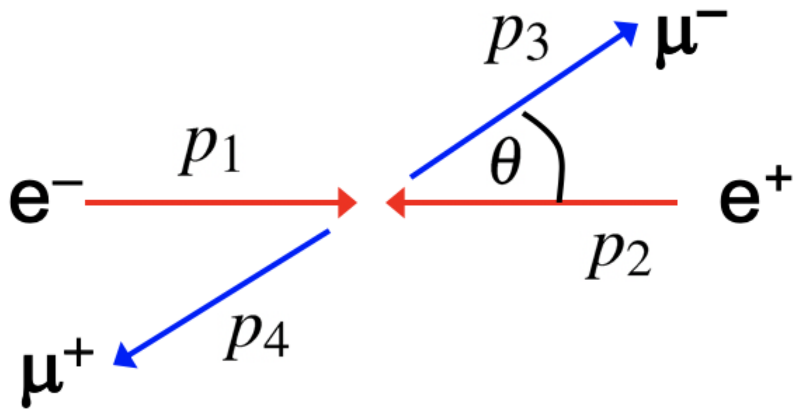
   - for scattering in the center-of-mass frame:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{\left|\vec{p}_f{}^*\right|}{\left|\vec{p}_i{}^*\right|} \left|M_{fi}\right|^2$$

   - for scattering in the lab. frame (neglecting the mass of the scattered particle):

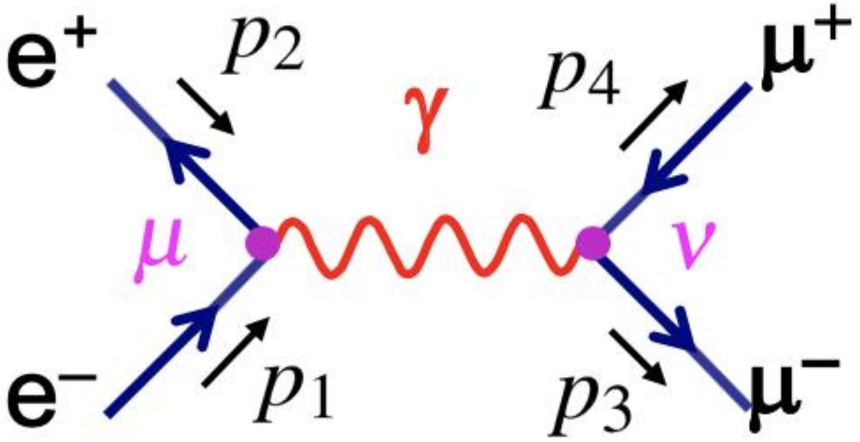
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 \left|M_{fi}\right|^2$$

# Electron-positron annihilation



Consider the process: $e^+e^- \to \mu^+\mu^-$

- We will work in CoM frame (appropriate for most $e^+e^-$ colliders)
  - $p_1 = (E, 0, 0, p), p_2 = (E, 0, 0, -p), p_3 = (E, \vec{p}_f), p_4 = (E, -\vec{p}_f)$
  - only consider the lowest order Feynman diagram



$$-iM = [\bar{v}(p_2) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3) ie\gamma^\nu v(p_4)]$$

- incoming anti-particle $\bar{v}(p_2)$
- incoming particle $u(p_1)$
- adjoint spinor written first

*Mnemonic rule:* the spinor representing the particle gong "away" from the vertex appears as the adjoint spinor

# Electron-positron annihilation

- In the CoM frame:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_f^{\;*}\right|}{\left|\vec{p}_i^{\;*}\right|}\left|M_{fi}\right|^2 \text{ with } s = (p_1 + p_2)^2 = (E + E)^2 = 4E^2$$

- Here $q^2 = (p_1 + p_2)^2 = s$

$$-iM = [\bar{v}(p_2) ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3) ie\gamma^\nu v(p_4)]$$

$$\Downarrow$$

$$M = -\frac{e^2}{s} g_{\mu\nu}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\nu v(p_4)]$$

# Electron and muon currents

- Previously we introduced the four-vector current:

$$j^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi$$

$j^{\mu}$ has the same form as the two terms in the brackets of the matrix element

- The matrix element can be written in terms of the $e$ and $\mu$ currents:

$$j^{\mu}_{(e)} = \bar{v}(p_2)\gamma^{\mu}u(p_1) \;\; \text{and} \quad j^{\nu}_{(\mu)} = \bar{u}(p_3)\gamma^{\nu}v(p_4)$$

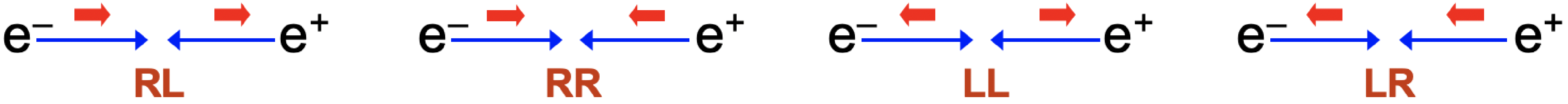
$$\Downarrow$$

$$M = -\frac{e^2}{s} g_{\mu\nu} j^{\mu}_{(e)} j^{\nu}_{(\mu)} = -\frac{e^2}{s} j_{(e)} \cdot j_{(\mu)}$$

- The matrix element is a scalar product of four-vectors ⟹ Lorentz-invariant quantity

# Spin in $e^+e^-$ annihilation

- In general, the electron and positron are not polarized ⟹ equal numbers of positive and negative helicity states
- Four possible combinations of spins in the initial state



- Similarly, there are four possible helicity combinations in the final state
- In total we get **16** orthogonal helicity combinations: **RL → RR, RL → RL, …**
- Each helicity combination corresponds to a separate physical process

# Spin in $e^+e^-$ annihilation

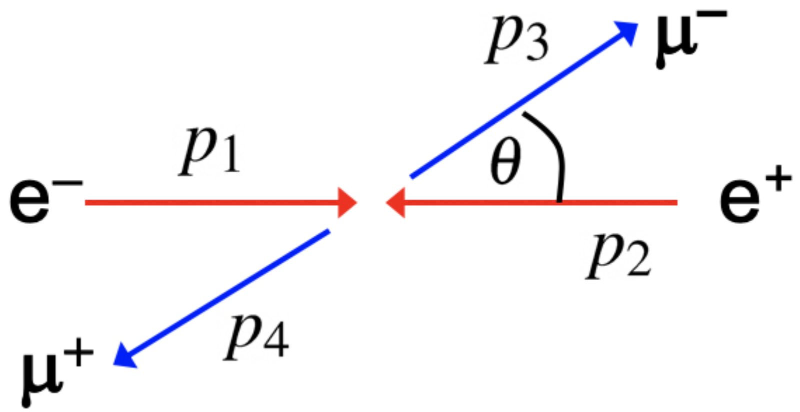
- To account for these states, we must:
  - sum over all 16 possible helicity combinations
  - average over the number of initial helicity states

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left( |M_{LL \to LL}|^2 + |M_{LL \to LR}|^2 + \cdots \right)$$

- We need to evaluate $M = -\frac{e^2}{s} j_{(e)} \cdot j_{(\mu)}$ for all 16 helicity combinations
- Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements
  - important feature of QED/QCD

# Spin in $e^+e^-$ annihilation

- In the CoM frame in the limit $E \gg m$
  - $p_1 = (E, 0, 0, E)$
  - $p_2 = (E, 0, 0, -E)$
  - $p_3 = (E, E\sin\theta, 0, E\cos\theta)$
  - $p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$



- Left- and right-handed helicity spinors for particles and antiparticles:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}, \quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

Using the shorthand notation $s = \sin\theta/2$ , $c = \cos\theta/2$ and $N = \sqrt{E+m} \approx \sqrt{E}$

# Spin in $\boldsymbol{e^+e^-}$ annihilation

- The initial-state $e^-$ can either be in a left- or right-handed helicity state

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

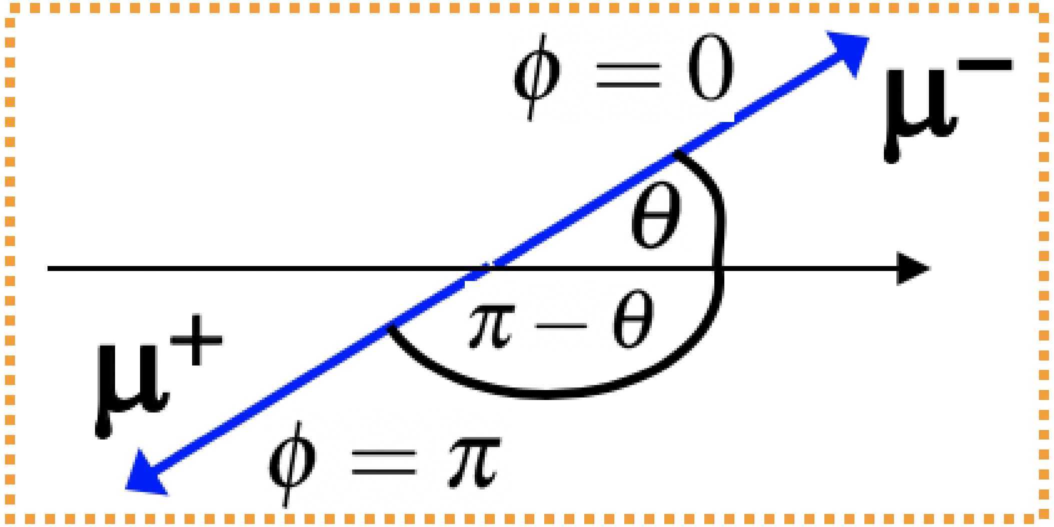
- The initial state positron ($\theta = \pi$) can have

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \qquad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

# Spin in $e^+e^-$ annihilation

- Similarly for the final state $\mu^-$ with a polar angle $\theta$ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \qquad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}$$



# Spin in $e^+e^-$ annihilation

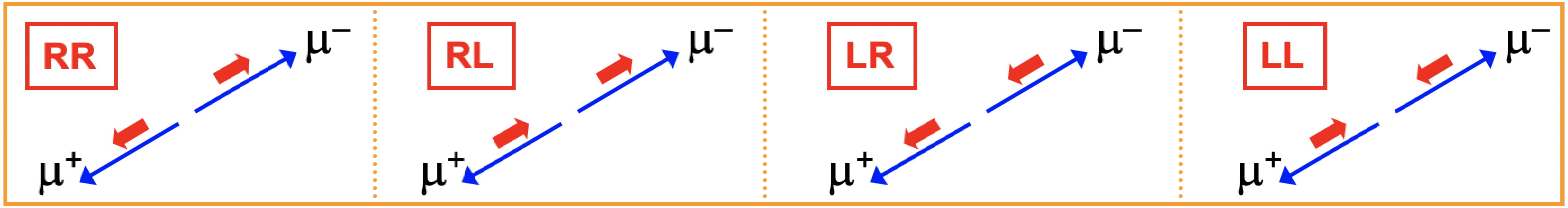
- And for the final-state $\mu^+$ replacing $\theta \to \pi - \theta, \phi \to \pi$ obtain:

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}, \qquad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

- Using $\sin\left(\frac{\pi-\theta}{2}\right) = \cos\theta/2$ and $\cos\left(\frac{\pi-\theta}{2}\right) = \sin\theta/2$, $e^{-i\pi} = -1$

- We want to calculate the matrix element $M = -\frac{e^2}{s} j_{(e)} \cdot j_{(\mu)}$

- First consider the muon current $j_{(\mu)}$ for 4 possible helicity combination:



# The muon current

- We want to evaluate $j^{\nu}_{(\mu)} = \bar{u}(p_3)\gamma^{\nu}v(p_4)$ for all helicity combinations
- For arbitrary spinors $\psi$ and $\phi$, it is straightforward to show that the components of $\bar{\psi}\gamma^{\nu}\phi$ are:

$$\bar{\psi}\gamma^{0}\phi = \psi^{\dagger}\gamma^{0}\gamma^{0}\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$

$$\bar{\psi}\gamma^{1}\phi = \psi^{\dagger}\gamma^{0}\gamma^{1}\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$

$$\bar{\psi}\gamma^{2}\phi = \psi^{\dagger}\gamma^{0}\gamma^{2}\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1)$$

$$\bar{\psi}\gamma^{3}\phi = \psi^{\dagger}\gamma^{0}\gamma^{3}\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2$$

# The muon current

- Let's consider $\mu_R^- \mu_L^+$ combination using $\psi = u_\uparrow, \phi = v_\downarrow$ with

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \qquad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

$$\bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0$$

$$\bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta$$

$$\bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE$$

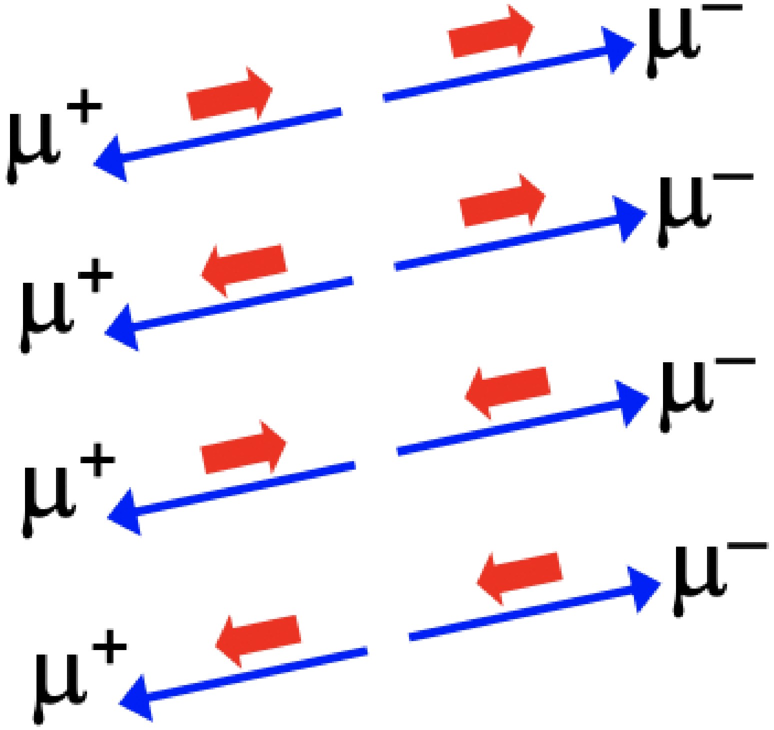
$$\bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta$$

# The muon current

- Giving for the four-vector muon current for the RL combination:

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

- The result for the four helicity combinations are:



$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) = (0,0,0,0)$$

$$\bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) = (0,0,0,0)$$

$$\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta)$$

# The muon current

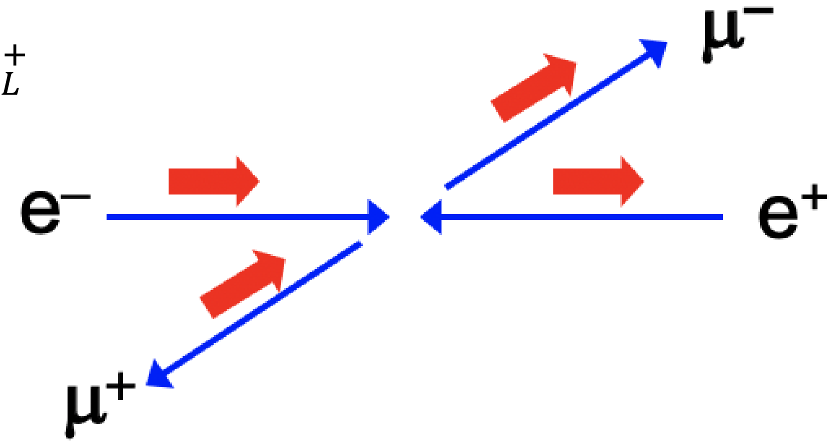
- **In the limit $E \gg m$ only two helicity combinations are non-zero!**
- Important feature of QED (applies also to QCD)
- In the weak interaction only one helicity combination contributes
- The origin of this effect will be discussed in the last part of this lecture
- As a consequence, of the 16 possible helicity combination, only four give non-zero matrix elements

# The muon current

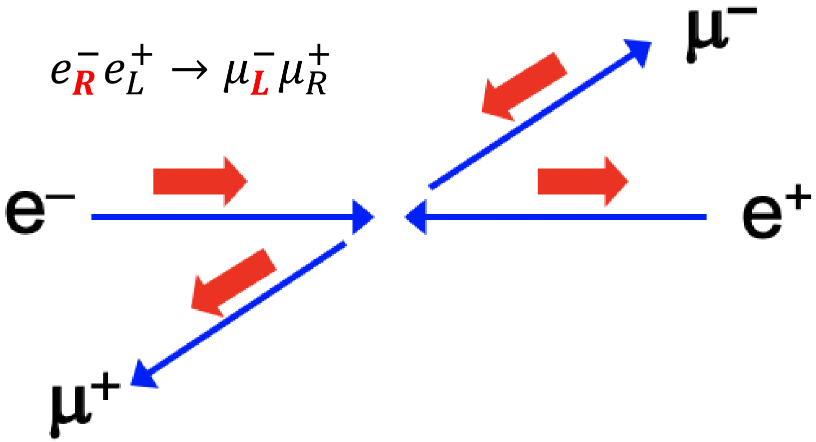
- For $e^+e^- \rightarrow \mu^+\mu^-$ now we only have to consider four matrix elements:

$e^-_R e^+_L \rightarrow \mu^-_R \mu^+_L$

$M_{RR}$



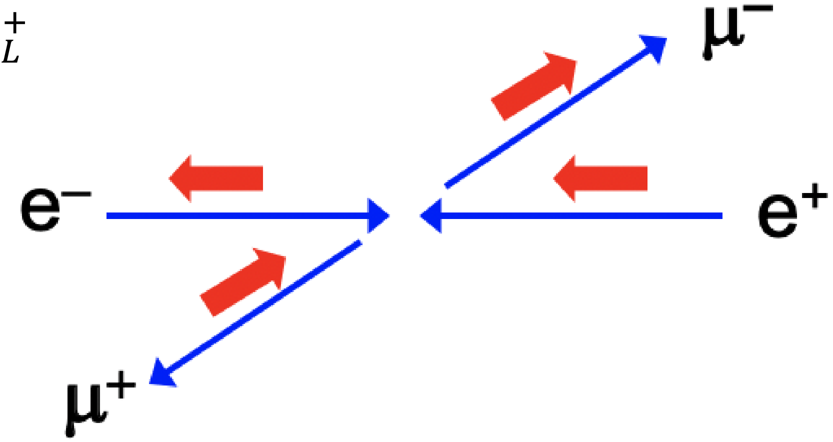
$e^-_R e^+_L \rightarrow \mu^-_L \mu^+_R$



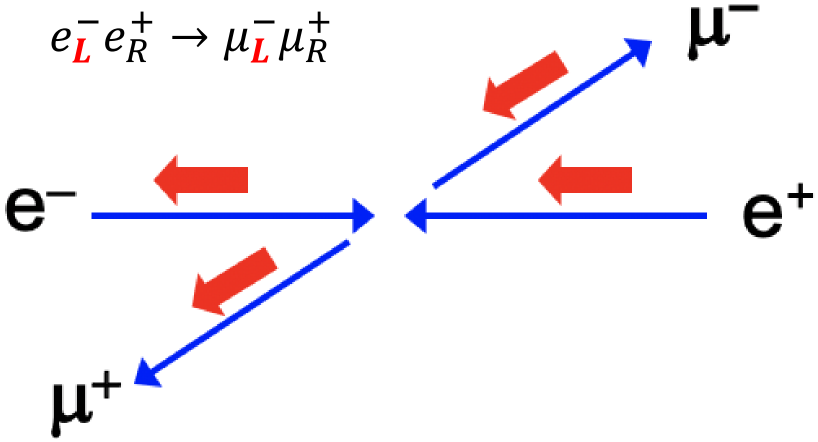
$M_{RL}$

$e^-_L e^+_R \rightarrow \mu^-_R \mu^+_L$

$M_{LR}$



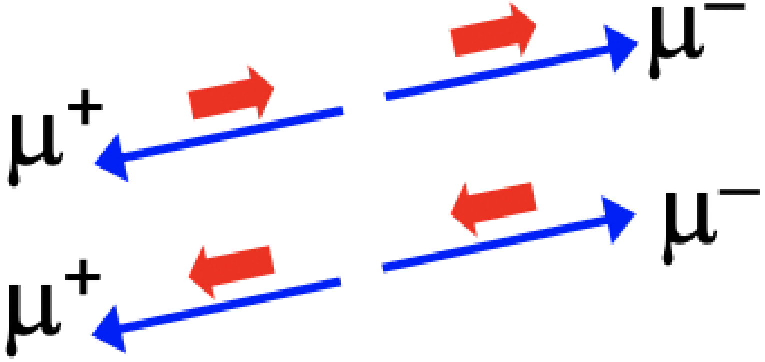
$e^-_L e^+_R \rightarrow \mu^-_L \mu^+_R$



$M_{LL}$

# The muon current

- We derived the muon currents for the allowed helicity combinations



$$\mu_R^-\mu_L^+: \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

$$\mu_L^-\mu_R^+: \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta)$$

- And we now need to focus on the electron current

# The electron current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \qquad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- The electron current can be obtained directly from the expressions for the muon current

$$j^{\mu}_{(e)} = \bar{v}(p_2)\gamma^{\mu}u(p_1) \text{ and} j^{\nu}_{(\mu)} = \bar{u}(p_3)\gamma^{\nu}v(p_4)$$

- Taking the Hermitian conjugate of the muon current gives:

$$[\bar{u}(p_3)\gamma^{\nu}v(p_4)]^{\dagger} = \left[u(p_3)^{\dagger}\gamma^{0}\gamma^{\nu}v(p_4)\right]^{\dagger}$$

$$= v(p_4)^{\dagger}\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3)$$

$$= v(p_4)^{\dagger}\gamma^{0}\gamma^{\nu}u(p_3)$$

$$= \bar{v}(p_4)\,\gamma^{\nu}u(p_3)$$

<u>Here we used:</u>

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}$$

$$\gamma^{0\dagger} = \gamma^{0}$$

$$\gamma^{\nu\dagger}\gamma^{0} = \gamma^{0}\gamma^{\nu}$$

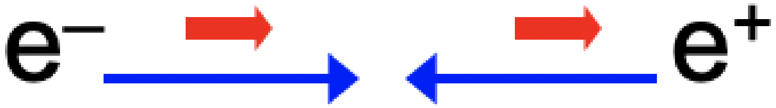
# The electron current

- Taking the complex conjugate of the muon currents for the two non-zero helicity configuration we get:

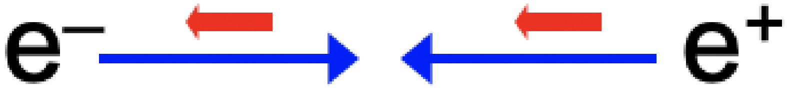
$$\bar{v}_\downarrow(p_4)\gamma^\nu u_\uparrow(p_3) = [\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)]^* = 2E(0, -\cos\theta, -i, \sin\theta)$$

$$\bar{v}_\uparrow(p_4)\gamma^\nu u_\downarrow(p_3) = [\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)]^* = 2E(0, -\cos\theta, i, \sin\theta)$$

- To obtain the electron currents we simply need to set $\theta = 0$



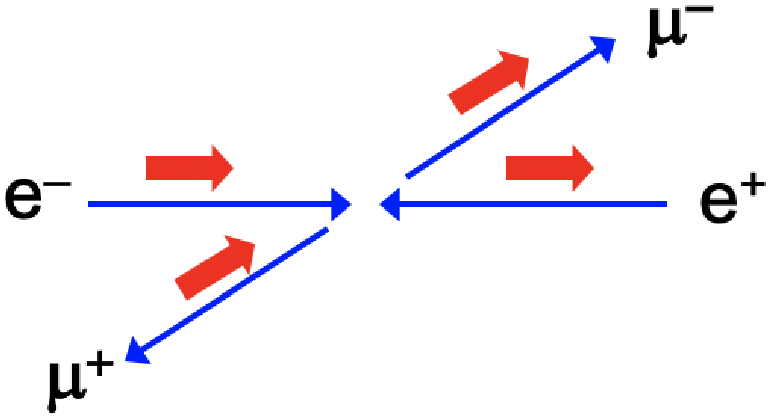
$e_R^- e_L^+ : \bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0, -1, -i, 0)$



$e_L^- e_R^+ : \bar{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1) = 2E(0, -1, i, 0)$

# Matrix element calculation

- We can now calculate $M = -\frac{e^2}{s} j_{(e)} \cdot j_{(\mu)}$ for the four possible helicity combinations
- e.g. we will do it for $e^-_{\boldsymbol{R}} e^+_L \to \mu^-_{\boldsymbol{R}} \mu^+_L$ which we will call $M_{\boldsymbol{RR}}$ (first subscript refers to the $e^-$ helicity and the second to the $\mu^-$ helicity and we don`t need to specify other helicities due to "helicity conservation")



$$e^-_R e^+_L : j^{\mu}_{(e)} = \bar{v}_{\downarrow}(p_2)\gamma^{\mu} u_{\uparrow}(p_1) = 2E(0, -1, -i, 0)$$

$$\mu^-_R \mu^+_L : j^{\nu}_{(\mu)} = \bar{v}_{\uparrow}(p_4)\gamma^{\nu} u_{\downarrow}(p_3) = 2E(0, -\cos\theta, i, \sin\theta)$$

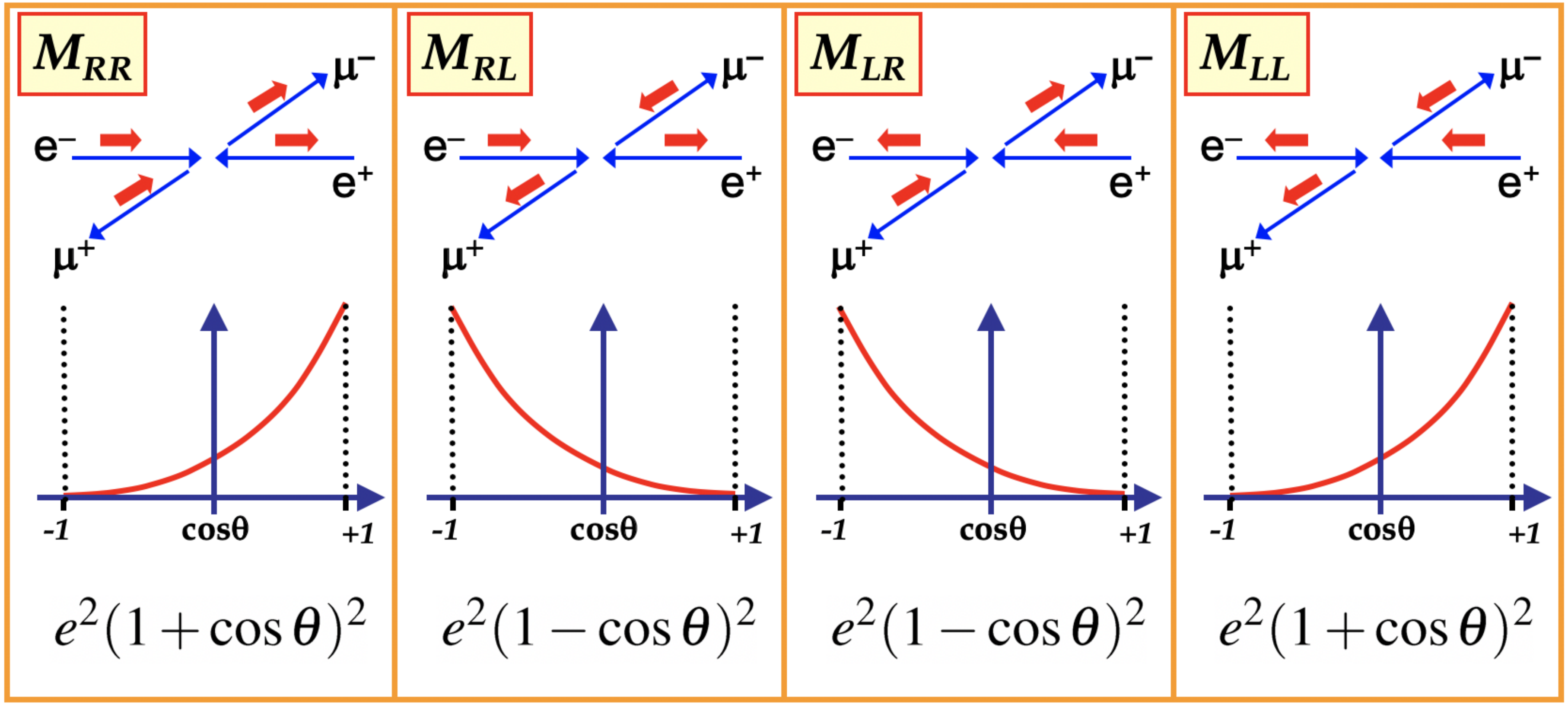
$$\Longrightarrow M_{RR} = -\frac{e^2}{s}[2E(0,-1,-i,0)] \cdot [2E(0,-\cos\theta, i, \sin\theta)] = -e^2(1+\cos\theta) = -4\pi\alpha(1+\cos\theta)$$

Here we used: $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$

# Matrix element calculation

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$

$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- Assuming that the incoming electrons and positrons are **unpolarized**, all 4 possible initial helicity states are equally likely (and in particular the 2 that contribute)

# Differential cross section

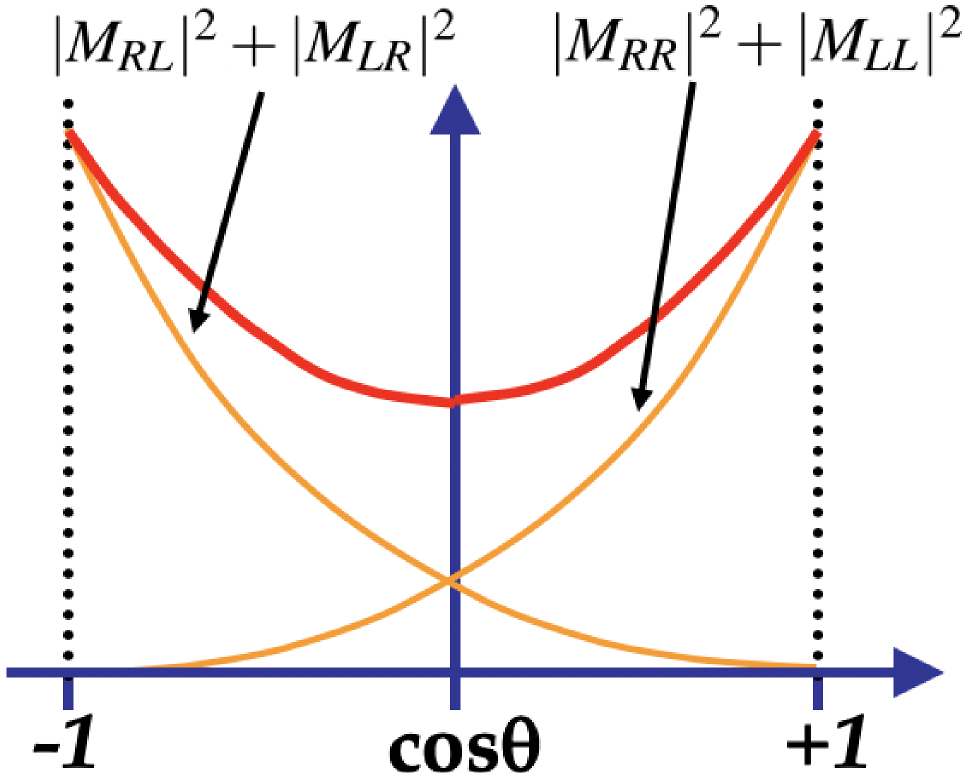
- Cross section: obtained by averaging over the initial spin states and summing over the final spin states

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{LR}|^2 + |M_{RL}|^2 + |M_{LL}|^2\right)$$

**Note:** no interference between amplitudes with different helicity configurations

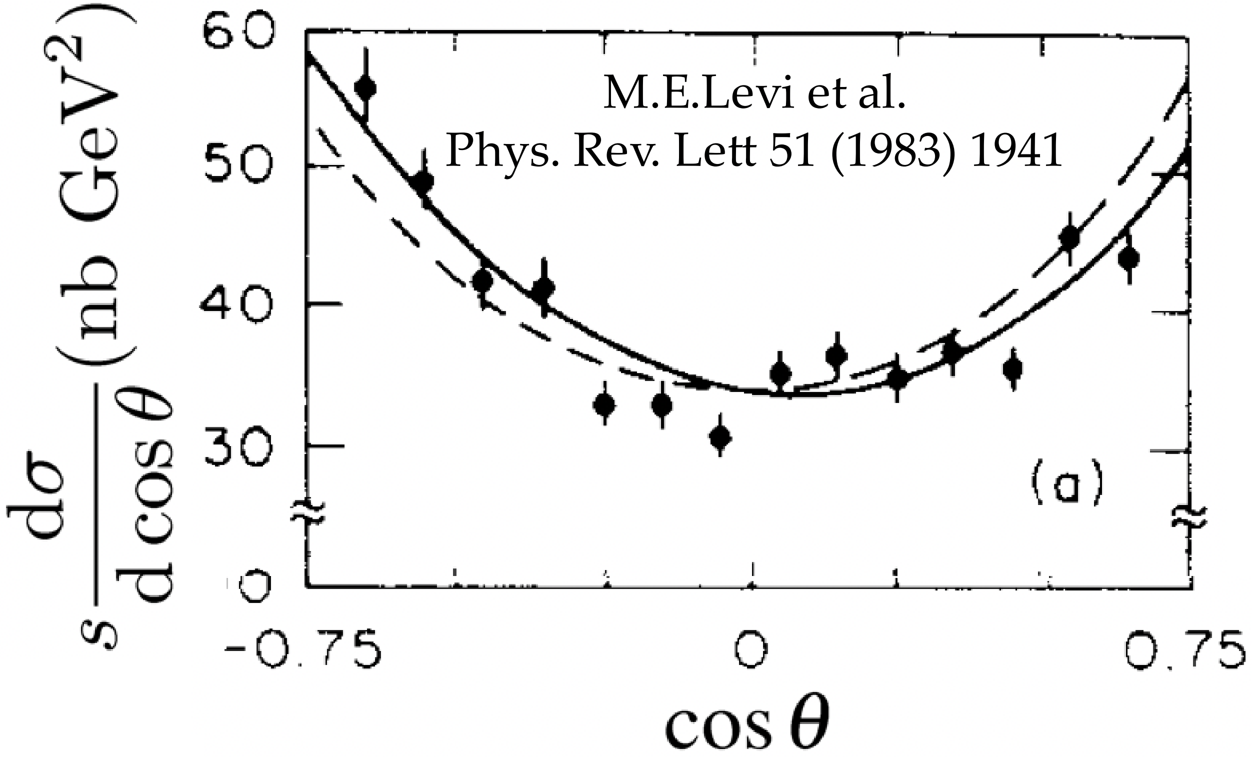
$$\frac{d\sigma}{d\Omega} = \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$



# Differential cross section: measurement

- Example: $e^+e^- \rightarrow \mu^+\mu^-$ at $\sqrt{s} = 29$ GeV
- Mark II experiment at the SLAC linear collider



M.E.Levi et al.
Phys. Rev. Lett 51 (1983) 1941

- - - - pure QED, $\mathcal{O}(\alpha^3)$
—— QED + Z contribution

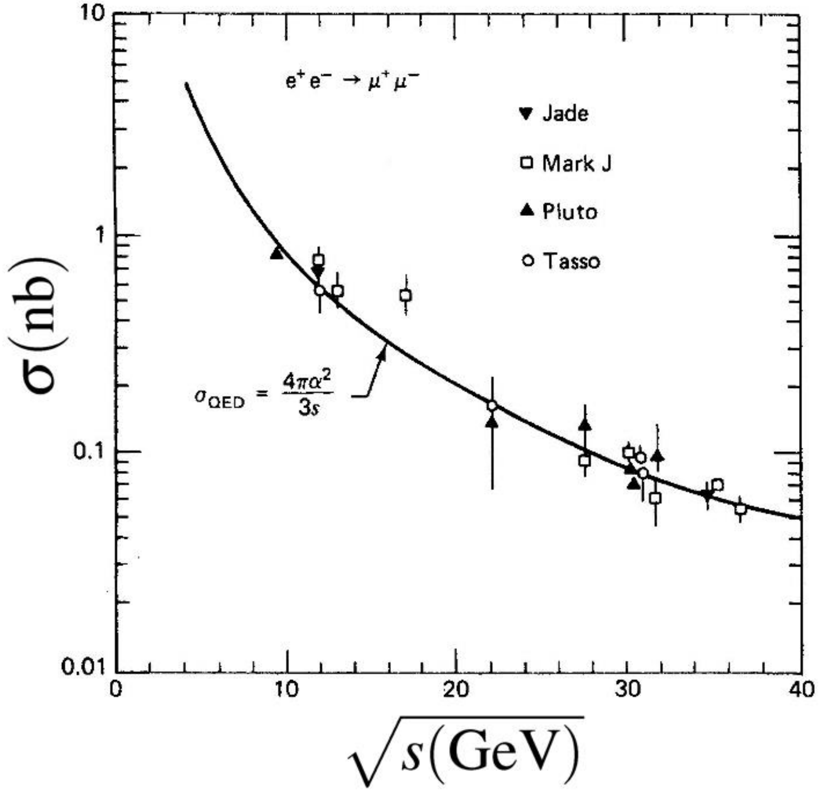
Angular distribution becomes slightly asymmetric in higher order QED or when Z distribution is included

# Total cross section: measurement

- The total cross section is obtained by integrating over $\theta, \phi$ using

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta) \quad \text{and} \int (1+\cos^2\theta)d\Omega = 2\pi\int_{-1}^{+1}(1+\cos^2\theta)d(\cos\theta) = \frac{16\pi}{3}$$

- We get the total QED cross section for the process $e^+e^- \rightarrow \mu^+\mu^-$

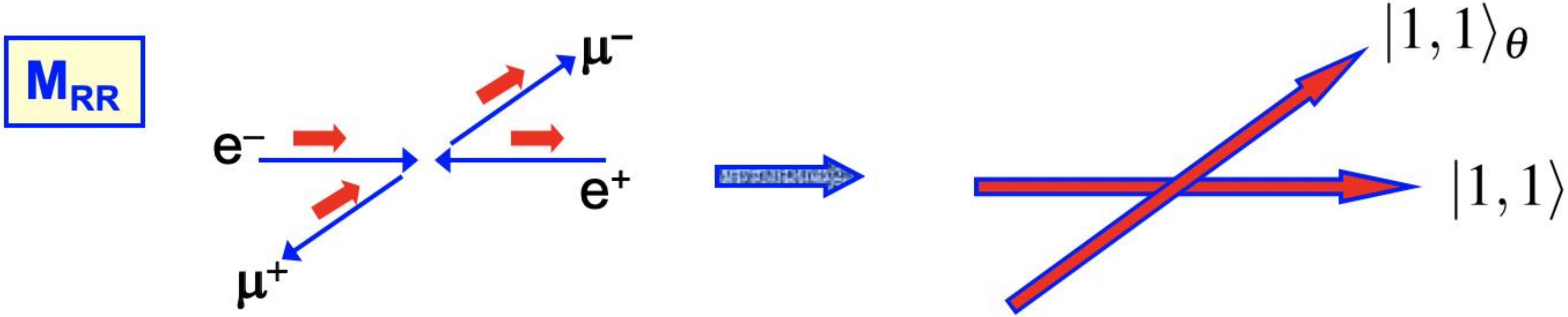


$$\sigma = \frac{4\pi\alpha^2}{3s}$$

- Lowest order cross section calculation provides a good description of the data
- This is an impressive result: from first principles we have arrived at an expression for the electron-positron annihilation cross section accurate to about 1% precision!

# Spin considerations ($\boldsymbol{E \gg m}$)

- The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$
- They are in a total spin$-1$ state, consistent with the exchange of a spin$-1$ particle, the photon, which is aligned along the $z$ axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly, the muon and anti-muon are produced in a total spin$-1$ state aligned along an axis with polar angle $\theta$



$\Rightarrow M_{RR} \propto \langle \psi | 1,1 \rangle$, where $\psi$ is the spin state, $|1, -1\rangle_\theta$ of the $\mu^+\mu^-$ pair

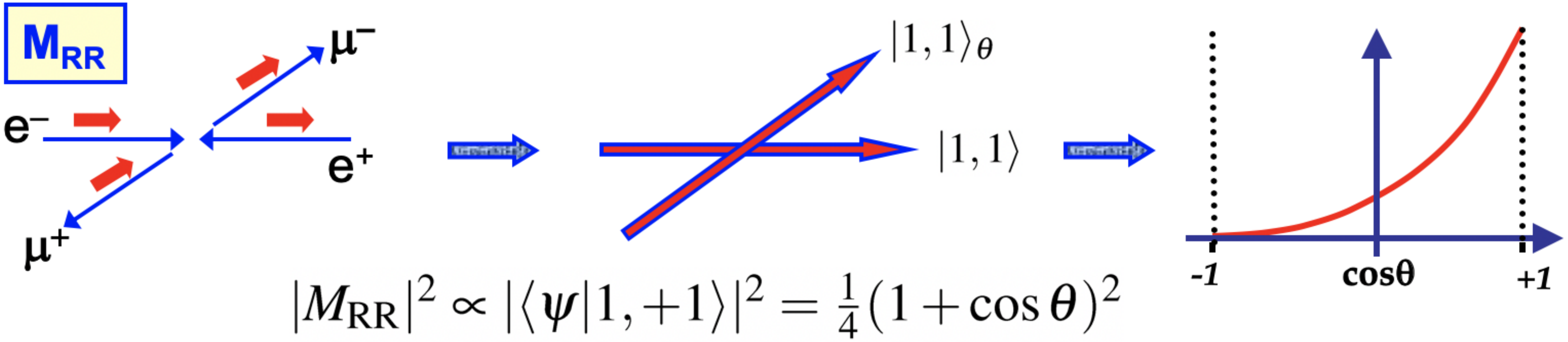
# Spin considerations ($\boldsymbol{E} \gg \boldsymbol{m}$)

- To evaluate this we need to express $|1,+1\rangle_\theta$ in terms of the eigenstates of $S_z$
- We can show that:

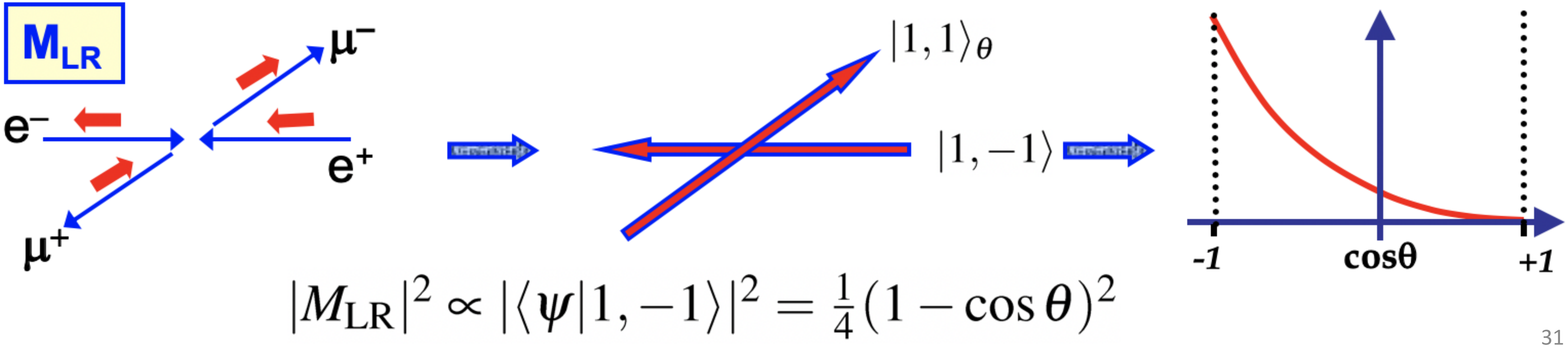
$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$$

# Spin considerations ($E \gg m$)

- Using the wavefunction for spin−1 state along an axis at an angle $\theta$ we can understand the angular dependence



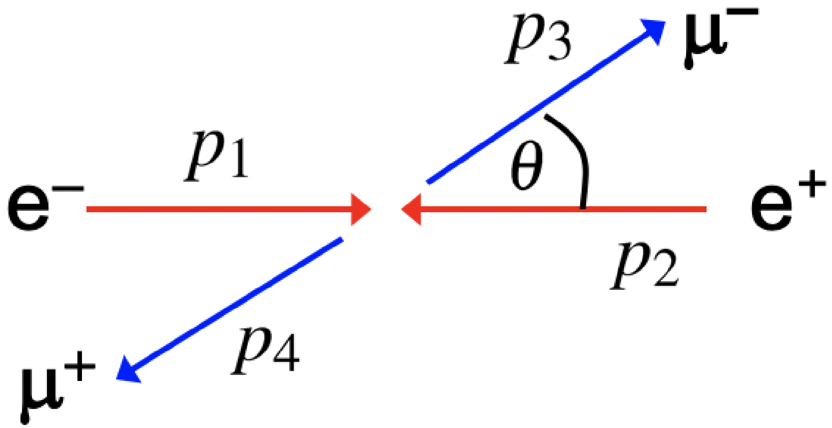
$$|M_{\mathrm{RR}}|^2 \propto |\langle \psi | 1, +1 \rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$



$$|M_{\mathrm{LR}}|^2 \propto |\langle \psi | 1, -1 \rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$

# Lorentz-invariant form of the Matrix Element

- Note that the derived spin-averaged ME is written in terms of the muon angle in the CoM frame



$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{LR}|^2 + |M_{RL}|^2 + |M_{LL}|^2\right) =$$

$$= \frac{1}{4} e^4 \left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right) \; = e^4 (1+\cos^2\theta)$$

- The matrix element is Lorentz-invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form (express in terms of 4-vector scalar products)

- In the CoM: $p_1 = (E, 0{,}0, E), p_2 = (E, 0{,}0, -E), p_3 = (E, E\sin\theta, 0, E\cos\theta), p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$

- From which we get: $p_1 \cdot p_2 = 2E^2$ , $p_1 \cdot p_3 = E^2(1-\cos\theta)$, and $p_1 \cdot p_4 = E^2(1+\cos\theta) \Longrightarrow$

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \frac{t^2 + u^2}{s^2}$$

# Chirality

- The helicity eigenstates for a particle/antiparticle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \qquad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}, \qquad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, \qquad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

Using $s = \sin\theta/2$ , $c = \cos\theta/2$ and $N = \sqrt{E+m} \approx \sqrt{E}$

- We can define the matrix:

$\gamma^5$ swaps components 1-3 and 2-4 of $\psi$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- In the limit $E \gg m$ (only in this limit) the helicity eigenstates are also eigenstates of $\gamma^5$

$$\gamma^5 u_\uparrow = +u_\uparrow, \qquad \gamma^5 u_\downarrow = -u_\downarrow; \quad \gamma^5 v_\uparrow = -v_\uparrow, \qquad \gamma^5 v_\downarrow = +v_\downarrow$$

# Chirality

- In general, we can define the eigenstates of $\gamma^5$ as left- and right-handed chiral states: $u_R, u_L, v_R, v_L$

$$\gamma^5 u_R = +u_R, \qquad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R, \qquad \gamma^5 v_L = +v_L$$

- In the limit $E \gg m$ (and only in this limit)

$$u_R \equiv u_\uparrow, \qquad u_L \equiv u_\downarrow, \qquad v_R \equiv v_\uparrow, \qquad v_L \equiv v_\downarrow$$

- **Important point:** in the general case, the helicity and chiral eigenstates are **NOT the same**!
- Only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates
- Chirality is an important concept in the structure of QED and any interaction of the form $\boldsymbol{\bar{u}\gamma^{\nu}u}$

# Chirality

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R, \qquad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R, \qquad \gamma^5 v_L = +v_L$$

- Define the projection operators:

$$P_R = \frac{1}{2}(1+\gamma^5), \qquad P_L = \frac{1}{2}(1-\gamma^5)$$

- The projection operators project out the chiral eigenstates

$$\begin{array}{llll} P_R u_R = u_R, & P_R u_L = 0\ , & P_L u_R = 0\ , & P_L u_L = u_L \\ P_R v_R = 0\ , & P_R v_L = v_L, & P_L v_R = v_R, & P_L v_L = 0 \end{array}$$

- *Note:* $P_R$ projects out right-handed particle states and left-handed antiparticle states

- We can then write any spinor in terms of its left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \frac{1}{2}(1+\gamma^5)\psi + \frac{1}{2}(1-\gamma^5)\psi$$

# Chirality in QED

- In QED the basic interaction between a fermion and a photon is given by

$$ie\bar{\psi}\gamma^{\mu}\phi$$

- We can decompose the spinors in terms of their Left- and Right-handed chiral components

$$ie\bar{\psi}\gamma^{\mu}\phi = ie(\bar{\psi}_L + \bar{\psi}_R)\gamma^{\mu}(\phi_L + \phi_R) =$$

$$= ie(\bar{\psi}_R\gamma^{\mu}\phi_R + \bar{\psi}_L\gamma^{\mu}\phi_R + \bar{\psi}_R\gamma^{\mu}\phi_L + \bar{\psi}_L\gamma^{\mu}\phi_L)$$

- Using the properties of $\gamma^5$:

$$(\gamma^5)^2 = I, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5$$

we can directly get

$$\bar{\psi}_L\gamma^{\mu}\phi_R = \bar{\psi}_R\gamma^{\mu}\phi_L = 0 \Longrightarrow ie\bar{\psi}\gamma^{\mu}\phi = ie(\bar{\psi}_R\gamma^{\mu}\phi_R + \bar{\psi}_L\gamma^{\mu}\phi_L)$$

- Only certain combinations of chiral eigenstates contribute to the interaction (**always** a true statement)

# Chirality in QED

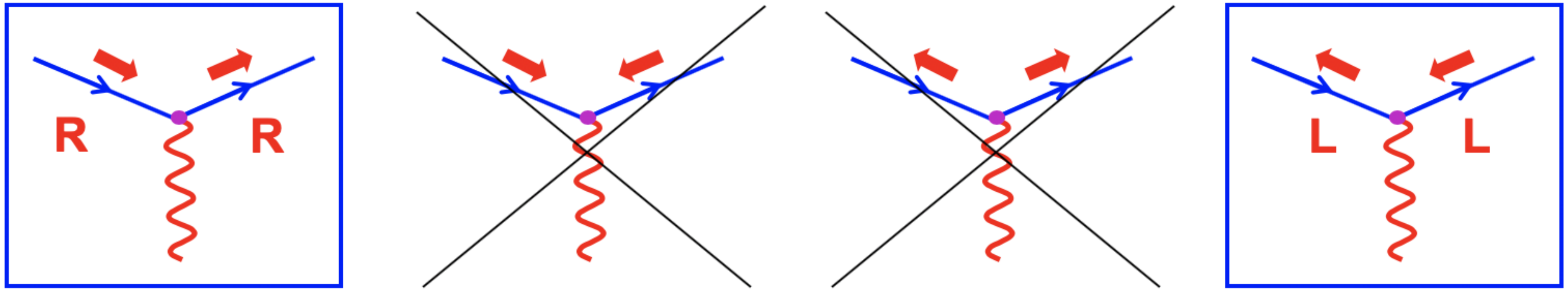
- For $E \gg m$ the chiral and helicity eigenstates are equivalent
- $\Longrightarrow$ for $E \gg m$ only certain helicity combinations contribute to the QED vertex
- This is why we previously found that for two of the four helicity combinations for the muons, the current were zero

# Allowed QED helicity combinations

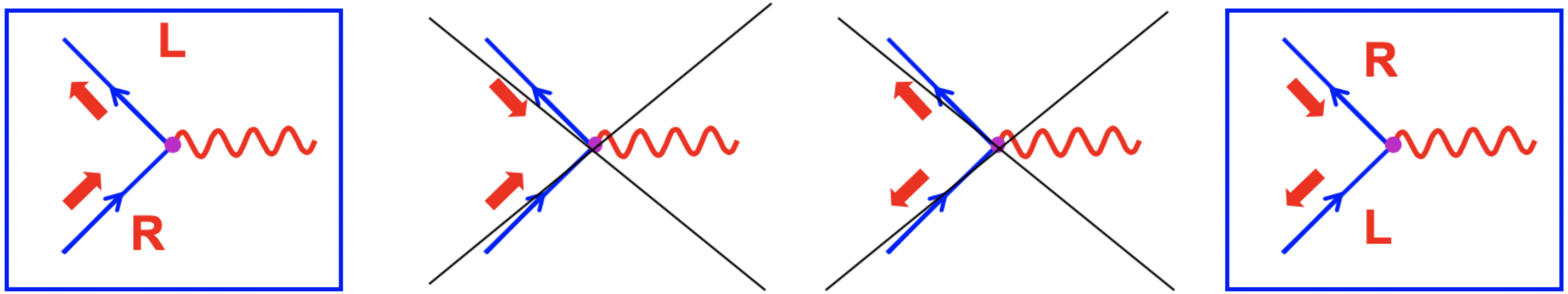
- In the ultra-relativistic limit, the helicity eigenstate ≡ chiral eigenstate
- In this limit, the only non-zero helicity combinations in QED are:

**Scattering:**

“Helicity conservation”



**Annihilation:**



# Summary

- In the center-of-mass frame the $e^+e^- \to \mu^+\mu^-$ differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta)$$

- Note: masses of the muons neglected $(E \gg m_\mu)$

- In QED only certain combinations of left- and right-handed chiral states have a non-zero matrix element

- Chiral states are defined by the chiral projection operators

$$P_R = \frac{1}{2}(1 + \gamma^5), \qquad P_L = \frac{1}{2}(1 - \gamma^5)$$

# Summary of Lecture 8

Main learning outcomes

- basic ingredients of a QED calculations
- derivation of the cross section for $e^+e^- \to \mu^+\mu^-$ annihilation as an example of calculation in QED taking into account the helicity of the electrons and muons
- correspondence between helicity and chirality